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Combating Gerrymandering with Ranked Choice Voting: An Experimental Analysis of Multimember Districts in the United States

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Abstract. Every representative democracy must specify a mechanism under which voters choose their representatives. The most common mechanism in the United States—winner-take-all single-member districts—both enables substantial partisan gerrymandering and constrains fair redistricting, preventing proportional representation in legislatures. We study the design of multimember districts (MMDs), in which each district elects multiple representatives, potentially through a non–winner-take-all voting rule. We carry out large-scale empirical analyses for the U.S. House of Representatives under MMDs with different social choice functions and algorithmically generated maps optimized for either partisan benefit or proportionality. Doing so requires efficiently incorporating predicted partisan outcomes—under various multiwinner social choice functions—into an algorithm that optimizes over an ensemble of maps. We find that, with three-member districts using single transferable vote, fairness-minded independent commissions would be able to achieve proportional outcomes in every state up to rounding, and advantage-seeking partisans would have their power to gerrymander significantly curtailed. Simultaneously, such districts would preserve geographic cohesion. Through simulation, we find that the insights are robust to cross-party voting. In the process, we advance a rich research agenda at the intersection of social choice and computational gerrymandering.

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1. Introduction

> This bill requires (1) that ranked choice voting … be used for all elections for Members of the House of Representatives, (2) that states entitled to six or more Representatives establish districts such that three to five Representatives are elected from each district, and (3) that states entitled to fewer than six Representatives elect all Representatives on an at-large basis—Fair Representation Act (2024).

The Fair Representation Act, first introduced in 2017 and reintroduced since,[1] would mandate the use of multimember districts (MMDs) to elect members to the U.S. House of Representatives, that is, having fewer, larger districts each with multiple representatives. The American Academy of Arts and Sciences (2020) released a report advocating for states to use multimember districts, however, "on the condition that they adopt a non-winner-take-all election model." Despite the popular focus on single-member district (SMD) elections, such MMDs have a long history in the United States, especially at the state and local levels. In 1962, 41 state legislatures had MMDs, often with winner-take-all models (Niemi et al. 1991); as of 2021, 10 state legislatures elect representatives for at least one chamber in such a manner. Arizona, for example, has two-member districts, in which each voter is given two votes and the top two vote-getters in each district are elected (Ballotpedia 2021a, b), and there is a debate in Maryland about whether to keep its (winner-take-all) MMDs. City councils, state parties, and other organizations often adopt more sophisticated techniques, using variations on ranked choice voting (RCV) to elect multiple winners from each of several districts (FairVote 2021).

This paper considers the design of such multimember districts and, in the process, advances a rich research agenda at the intersection of two well-studied aspects of the design of representative democracies: partisan gerrymandering and social choice for multiple winners. The gerrymandering literature largely (but not exclusively) assumes winner-take-all voting in SMDs and, given a fixed set of voters, studies how to divide the voters into districts such that the set of winners across districts satisfies desirable properties (e.g., for a gerrymandering party, maximizing the number of winners belonging to their party; for a neutral rule maker, devising rules to ensure proportionality such that the fraction of winners of each party matches the fraction of voters). The theoretical social choice literature, on the other hand, primarily considers a single district and studies voting rules (functions from voter rankings to a set of winners) such that the set of winners in that district satisfies similar properties, including proportionality. A map consisting of multiple MMDs (e.g., 30 two-member districts in Arizona) requires both partitioning voters into districts and devising rules for how each district collects and aggregates votes. The challenge cannot be decomposed: effectively drawing districts depends on the social choice function, and the same social choice function has different effects depending on district size and composition.

We study this joint challenge, carrying out large-scale empirical analyses for the U.S. House of Representatives under MMDs with different social choice functions under maps that could be drawn by either partisan gerrymanderers or independent commissions. We show that, indeed, the choices of (how many and which) districts to draw and which voting method to use should not be made independently. For example, whereas MMDs with winner-take-all elections are commonly believed to be discriminatory against minorities when compared with SMDs (Derfner 1972, Niemi et al. 1985, Bullock and Gaddie 1993), we empirically find that MMDs with more proportional voting methods support political minorities. We further find that interior solutions in this joint optimization—with neither SMDs nor one large MMD—often best balance the multiple objectives promoted by good governance groups, such as proportionality, competitiveness, and geographic district size. In more detail, our contributions and findings are as follows.

1.1. Methodological Contributions

We provide a scalable, empirical methodology to algorithmically study partisan gerrymandering and fair redistricting under multimember districts. We leverage an efficient way to calculate partisan outcomes under single transferable vote (STV; a common variant of RCV for multiple winners) and extend an algorithm by Gurnee and Shmoys (2021) to efficiently calculate near-optimal multimember maps under a variety of objective functions and social choice rules. We further develop an individual-level, voter file–based methodology to study party crossover and intraparty effects of ranked choice voting. Our results, using voting and individual-level data from across the United States, further illustrate how theoretical social choice guarantees heterogeneously translate to practice away from the asymptotic regimes in which they are often studied. We believe that this work and methodological approach can be applied beyond the context of multimember districts for legislatures in the United States for questions regarding the design of representative democracies at the intersection of gerrymandering and social choice across computer science, optimization, and political science.

1.2. Application-Oriented Findings

We apply our methods to consider the design of MMDs empirically in the House of Representatives in the United States. We primarily study proportionality[2] as a function of the number of districts and the voting rule used. Surprisingly, our results indicate that two- or three-member districts with proportional voting rules are often sufficient to prevent the worst gerrymanders by an adversary and to enable independent commissions to find a proportional map. Even small MMDs are enough to eliminate natural gerrymandering, in which geography and the distribution of citizens lead to a natural advantage for one party, and purposeful ones. This result holds with no other optimization constraints besides district contiguity and continues to hold when voters behave noisily and potentially vote across parties. On the other hand, we find that poor design constraints (in either the number of districts or the voting rule used) enable more extreme gerrymanders than are possible with SMDs.

These effects occur because MMDs with appropriate social choice functions protect concentrated (political) minorities by making packing and cracking more difficult and support diffuse minorities by joining them in the same large multimember district with a lower threshold to win a seat. To underscore the practical relevance of our methods, we analyze the specific proposal in the Fair Representation Act and show that it achieves an effective balance in the MMD design space though, as emphasized above, its prescribed three- to five-member districts are larger than are necessary; this part of the proposal could be relaxed without sacrificing its proportionality goals.

Finally, in the Online Appendix, we study how MMDs affect intraparty diversity, measuring two competing claims: that small SMDs protect diverse, geographically correlated interests within a party (e.g., urban, Black Democrats versus suburban, white Democrats) and, conversely, that RCV with large MMDs enables ideological differences within a party (cf. Page and Gilens 2018, FairVote 2021). We find, under reasonable assumptions on how voters rank candidates within a party, that two- or three-member districts preserve geographic compactness and, thus, the notion that legislators represent a geographically cohesive set of voters, increasing the partisan diversity of the winning set and, thus, striking a balance between the competing claims.

Section 2 presents the model and methods. Section 3 contains our main results regarding interparty effects of multimember districts. Section 4 extends the results to a setting in which voters behave noisily and may vote across parties. We conclude in Section 5. Online Appendix B considers intraparty effects. Our code is available at https://github.com/nikhgarg/gerrymandering_mmd.

1.3. Related Work

This work is at the intersection of two rich literatures, showing how joint optimization over how voters are split into districts (gerrymandering) and how they vote within a district (social choice) yields outcomes that neither could achieve separately. It further contributes to the literature studying multiwinner districts in the United States. Given each field's vast literature, we focus on the most relevant work.

1.3.1. Redistricting. Gerrymandering is the practice of using district boundaries to engineer electoral outcomes by packing and cracking voters within and between districts (Erikson 1972, Issacharoff 2002, McGann et al. 2016). Because of the status quo rules in the United States, past computational research primarily studies gerrymandering and redistricting more broadly in the context of a fixed number of single-member districts with winner-take-all elections.

Optimal political districting is well known to be an NP-hard computational problem (Dyer and Frieze 1985, Puppe and Tasnádi 2008, Van Bevern et al. 2015, Lewenberg et al. 2017, Kueng et al. 2019, Chatterjee and DasGupta 2020). Therefore, to study redistricting, researchers have developed ensemble methods (Chen and Rodden 2013; Liu et al. 2016; Autry et al. 2020, 2021; Fifield et al. 2020; DeFord et al. 2021; Gurnee and Shmoys 2021; McCartan and Imai 2023) that generate huge quantities of legal district plans to explore the exponentially large space of feasible maps.

Such techniques are commonly used to detect gerrymandering (Chikina et al. 2017; Herschlag et al. 2017, 2020; Duchin 2018; Duchin et al. 2022) or to study the impact of different redistricting rules on the distribution of outcomes (DeFord and Duchin 2019, DeFord et al. 2020). An important result is that even under neutral maps drawn without regard to underlying political geography, the natural geographic segregation among partisan voters can create skewed political outcomes (Chen and Rodden 2013, Borodin et al. 2022). More recently, Duchin and Schoenbach (2022) find that plurality single-member districts are compatible with proportionality in many states if used as a target for redistricting.

There are many proposed metrics of fairness in redistricting, including the efficiency gap (Stephanopoulos and McGhee 2015), the mean-median gap (Wang 2016), partisan-symmetry (Warrington 2018), competitiveness (DeFord et al. 2020), and, most simply, proportionality. Such metrics can be used as the objective function of an optimization algorithm to generate district maps with (un)fair outcomes (King et al. 2015, Gurnee and Shmoys 2021, Cannon et al. 2023, Swamy et al. 2023). However, because of partisan segregation, in some states, creating a proportional plan with SMDs is actually impossible (Duchin et al. 2019), motivating the use of alternative election procedures. In this work, we primarily consider proportionality as our fairness notion.

Methods to restrict gerrymandering are also commonly proposed. Borodin et al. (2022) find that compactness constraints do not effectively prevent gerrymandering. Inspired by the fair division literature, researchers have also recently proposed methods in which two adversarial parties together draw a map (Pegden et al. 2017, Tucker-Foltz 2018, Benade and Procaccia 2020, Benade et al. 2026).

1.3.2. Social Choice. The social choice literature, by contrast, typically fixes a set of voters and a number of winners and studies how voting rules (how preferences are elicited and aggregated) affect outcomes (Arrow et al. 2010, Dewan and Shepsle 2011, Brandt et al. 2016). Here, we focus on the literature on multiwinner elections. The theoretical social choice literature considers properties of voting rules, often for arbitrary distributions of voters (Conitzer and Xia 2012; Aziz et al. 2015, 2017; Caragiannis et al. 2017; Elkind et al. 2017a; Fishburn and Pekeč 2018; Cheng et al. 2019; Garg et al. 2019; Lackner and Skowron 2019). Most relevant is Skowron (2021), who studies the proportionality guarantees of various multiwinner voting rules, including Thiele rules, finding proportional approval voting (PAV; a type of Thiele rule) to be the most proportional with respect to their measure though other Thiele methods may outperform it on objectives beyond proportionality. The optimality of PAV and the parameterizable nature of the class motivates us to focus on Thiele rules in this work, and our results complement the theoretical work in comparing such rules in nonasymptotic settings with empirical voter distributions. Related to our intraparty analysis, Elkind et al. (2017b) simulate the properties of multiwinner voting rules (such as STV and PAV) when voters lie on a two-dimensional Euclidean space.

On the applied side, a strand of the literature studies the effects of voting reforms, especially on minority voters. McDaniel (2018) finds that RCV increased racially polarized voting in comparison with runoff elections. McGhee and Shor (2017) find mixed evidence for whether top-two primaries have reduced polarization, and Rogowski and Langella (2015) find no evidence that open versus closed primaries affect candidate ideology; Grose (2020) finds the opposite on both points. Spencer et al. (2015) advocate for ranked choice voting as a Voting Rights Act remedy, supporting its legality.

1.3.3. Multimember Districts in the United States. As chronicled by Klain (1955), there is a rich history of multimember districts in the United States in state legislatures and city councils. In 1962, for example, 30 states used MMDs to elect state senators, and 41 states used them to elect state representatives, mostly in two- to four-member districts with winner-take-all procedures (Niemi et al. 1991). Historically, at-large city council members have often also been elected with equivalent procedures. On the other hand, since 1967, federal law has banned the use of MMDs to elect members to the U.S. House of Representatives, a status quo the Fair Representation Act would repeal.

The political science community studies these districts, often focusing on their effect on race (Silva 1964, Banzhaf 1966, Dauer 1966) though some have questioned the optimal district size (Hamilton 1967). In particular, whereas the empirical evidence is mixed because of the observational nature of analysis, the predominant view is that winner-take-all MMDs harm racial minorities, especially in Southern states (Derfner 1972, Niemi et al. 1985, Bullock and Gaddie 1993). Similarly, winner-take-all at-large city council elections are thought to dilute the votes of minorities (Davidson and Korbel 1981, Walawender 1985, Bullock 1989, Still 1991, Trounstine and Valdini 2008). As theoretical models (Gerber et al. 1998) elucidate, such districts with winner-take-all rules enable the majority—even without substantially distorting district lines as might be necessary with SMDs—to ensure that a minority, even with 49% of the vote, elects none of its preferred candidates. Partially because of such evidence, by 1982, the majority of states had eliminated the use of MMDs for their state legislatures, and several cities have eliminated their at-large council seats under court mandate. Lempert (2021) analyzes the legality of RCV and multimember districts and advocates for it as a court-ordered remedy to illegal gerrymanders.

Most related is work favorably comparing MMDs using ranked choice voting to SMDs for city councils. Benade et al. (2021) compare outcomes using RCV versus single-member districts in four empirical case studies, finding that it yields better representation for demographic minorities; key to their approach is constructing various ranking-based models for how voters prefer candidates beyond a "solid coalitions" assumption in which voters prefer same-party candidates before other-party ones. From the same group, Buck et al. (2019) and Angulu et al. (2019) propose using RCV with several MMDs instead of SMDs for the Lowell, Massachusetts, and Chicago, Illinois, city councils, respectively. Their work primarily considers nonpartisan representation, such as race and other demographic characteristics; these factors are more salient in municipal contexts, in which such MMDs with RCV are more common today. Section 4 and Online Appendix B explore similar questions with cross-party voting and intraparty effects, respectively, with complementary voter file–based methods. More recently in subsequent and complementary work, the MGGG Redistricting Lab (2022) also analyzes the effects of the Fair Representation Act; it considers a more limited set of district sizes and focuses on racial representation as opposed to our focus on partisan effects.

To this literature, our work adds a systematic analysis of the partisan gerrymandering effects of such districts: providing a scalable methodology, characterizing the effects on proportional representation under both adversarial gerrymandering and fair redistricting across the range of possible district sizes, and providing design recommendations for legislation. We further study how MMDs affect intraparty results on two dimensions: political preference and geography.

1.3.4. International Contexts and Other Political Solutions. Countries beyond the United States also use MMDs; in Singapore, for example, the ruling party has been accused of using MMDs with winner-take-all rules to gerrymander (with boundaries determined immediately before the election) (Tung 2020). Empirically comparing elections across 81 countries, Carey and Hix (2011) find that larger district sizes are correlated with higher proportionality; they find that multimember districts of size four to eight exhibit much of the proportionality benefits and argue for them as the "electoral sweet spot." Our work suggests that, in the United States, even smaller districts would suffice. Horwill (1925) and Taagepera and Shugart (1989) also highlight district size as an important factor in proportionality. Lijphart (1999) compares majority and proportional representation systems internationally and highlights that two-member districts (with plurality voting) have been used in the United Kingdom, Canada, India, and

Barbados but not since 1970; Mauritius has continued to use three-member districts, using plurality voting.

Multimember districts are far from the only possible governmental system to achieve proportionality or other related desiderata. Many countries have seat allocation systems designed to be proportional with multiple parties (see Pukelsheim 2017 for a survey). For example, party list systems (as in Israel and the Netherlands) (Lijphart 1999) allow voters to specify a party (potentially alongside specific candidates); each party above a threshold then receives an approximately proportional number of seats with methods such as Jefferson/D'Hondt used to convert proportions to integral seats. Such systems and related ones are also studied in the EconCS and operations communities (e.g., Cembrano et al. 2021a, b). Güney (2018) and Bredereck et al. (2020) analyze the margin of victory in proportional representation systems, and this is related to our competitiveness analysis for multimember districts.

Finally, we note that other reforms are also of interest in the United States with other proposals to achieve similar goals (e.g., Ernst 1994, Balinski and Young 2001). For example, Bachrach et al. (2016) consider the misrepresentation ratio of single-member districting with various score-based ranking rules, in Electoral College–type settings: when these rules are used to elect a winner within each district but then an overall winner is declared based on the majority vote of districts.

A comparative empirical analysis of such approaches—and especially which may be most politically feasible in various jurisdictions—is outside the scope of this work.

2. Model and Methods

Section 2.1 formalizes the joint redistricting and social choice challenge, Section 2.2 introduces the social choice functions we analyze and shows how to efficiently compute per-party seat shares under them, and Section 2.3 presents our empirical method.

2.1. Problem Definition

The task is to elect a legislature composed of N seats. There is a population of voters $\mathcal{V}$; each voter $v \in \mathcal{V}$ lives in an atomic block $b_v \in \mathcal{B}$ and belongs to a party $p_v \in \{R, D\}$. The blocks $\mathcal{B}$ are organized in an adjacency graph in which two blocks share an edge when they are geographically adjacent to each other.

2.1.1. Multimember Redistricting. In redistricting, the blocks $\mathcal{B}$ are partitioned into K geographically contiguous districts, in which each district k is allocated $N_k \in \mathbb{N}$ seats in the legislature such that $\sum_k N_k = N$ and $N_k \geq 1$. Each district also must be population-balanced in accordance with N_k such that the population ratio of district k to the whole state is bounded between $\frac{N_k \pm \epsilon}{N}$ for population tolerance ϵ.[3] The resulting set of districts is referred to as a map or a plan. Maps can be drawn to fulfill various objective functions. For example, a party can gerrymander a map to maximize the seats won by their party, or an independent commission can attempt to satisfy one or more notions of fairness.

2.1.2. Social Choice Function. Each district runs a separate election to fill its N_k seats with each voter v voting in the district $k(b_v)$ to which the voter's block b_v belongs. Each election is run according to a social choice function F that determines what information each voter provides (in this work, either a set of approved candidates or a ranking over candidates) and how votes are aggregated to produce the N_k winners in each district k.

Together, the choice of map (K contiguous districts with N_k seats each) and social choice function F define an election. Given data on voters and assumptions on how they rank candidates (detailed in Section 2.2), the procedure determines the set of N winners. (In our empirical optimization approach, we use a probabilistic approach to compute the expected seat share based on the variance of past elections to model the heterogeneous political elasticity and noise among different constituencies.)

2.1.3. Evaluation Metrics. An election procedure can be evaluated based on the set of winners it produces. To measure outcomes across parties, we primarily consider proportionality: how does the fraction of winners w_p belonging to each party $p \in \{R, D\}$ (seat share) compare with the fraction of voters y_p (vote share); the proportionality gap is the difference $|w_R - y_R|$. The larger the gap, the more that the procedure favors one party over another. In Online Appendix B, we further consider intraparty measures, that is, how the election procedure influences within-party differences.

2.1.4. Research Questions. We ask, how does the election procedure determine the induced outcomes, and what is the joint influence of the social choice function F and the map? Note that the various components of the election procedure may be determined by different actors with competing interests. The Fair Representation Act, for example, would mandate that $N_k \in \{3, 4, 5\}$ and an RCV-based social choice function be used. However, barring a separate mandate concerning independent commissions, in each state, a partisan state legislature may still seek to draw maps most favorable to their party.[4] Thus, we, in particular, are interested in the following design question: how do various constraints on the election procedure design space (restrictions on K, N_k, or F) affect the range of outcomes possible under maps drawn by either partisan actors or independent commissions?

2.2. Social Choice Functions, Voter Assumptions, and Calculating Seat Shares

We study our research questions empirically, analyzing outcomes under counterfactual election procedure designs. However, doing so requires converting historical voting data to expected electoral outcomes under various social choice functions and with hypothetical candidates. Further, outcomes must be efficiently computable as our redistricting optimization algorithm (introduced in Section 2.3) calculates, as a subroutine, the seat share under a given district and social choice function. We now lay the groundwork for our empirical method by introducing our social choice functions and assumptions and showing how to efficiently compute per-party seat shares in a district.

2.2.1. Social Choice Functions Considered. We consider two well-studied classes of multiwinner social choice functions: Thiele rules and STV. Variants of STV are used in elections in Ireland, Australia, Scotland, Malta, and locally in the United States (Tideman 1995, Bowler and Grofman 2000, FairVote 2021, Electoral Reform Society 2026). PAV, whereas not as widely used in practice, is well-studied in the literature because of its strong theoretical guarantees (Aziz et al. 2017, Lackner and Skowron 2019, Skowron 2021).

A Thiele rule is characterized by a function λ. Each voter v in district k provides a set S_v of the N_k candidates the voter likes most. The set of winners is determined as follows. Consider a potential set of winners C. The number of points that voter v contributes to C is $\sum_{i=1}^{|S_v \cap C|} \lambda(i)$, and the set C with the most points across voters (after tie-breaking) is selected. The (nonincreasing) function λ establishes that votes have diminishing returns; the ith approved candidate in the set is worth $\lambda(i)$. For example, winner-take-all approval voting is a Thiele rule with $\lambda(i) = 1$: for each voter, the set C receives a number of points equal to the number of candidates in the set that the voter likes, and so the N_k candidates with the most votes win. We further consider PAV, a Thiele rule with $\lambda_{\text{PAV}}(i) = \frac{1}{i}$, and Thiele squared with $\lambda_{\text{TS}} = \frac{1}{i^2}$. PAV is well-studied in the theoretical social choice literature and comes with optimality guarantees (in terms of proportionality) (Skowron 2021). Thiele squared induces even sharper diminishing returns than PAV in a voter having more of the voter's preferred candidates winning and, thus, prioritizes more voters having at least one approved candidate (and 50/50 outcomes between two parties regardless of underlying vote shares). More generally, Thiele rules both contain common voting rules and parameterize a rule's tendency to favor proportional or balanced outcomes.

In STV, a type of ranked choice voting for multiple winners, each voter instead submits a ranking over candidates (here, we assume the rankings are complete, not partial). Consider a district with N_k seats and V_k voters; the set of winners is created iteratively as follows. The number of first place votes for each candidate is counted. Any candidate with a number of votes at least the droop threshold $Q = \lfloor \frac{V_k}{N_k+1} \rfloor + 1$ is selected as a winner, and the surplus votes (number of votes minus Q) are transferred to each voter's next preference. If no candidate has enough votes, then the candidate with the least number of votes (with tie-breaking) is eliminated, and the candidate's votes are transferred. The process continues until N_k candidates are selected or the number of remaining candidates equals the number of remaining seats. STV is commonly used in practice in multiwinner elections.

Details can differ in how such votes are transferred, but our results in Section 3 do not depend on the transfer rule; in our simulations in Section 4 and Online Appendix B, we use the fractional STV rule, in which each voter keeps a fractional number of votes equal to the voter's share of the winning candidate's surplus votes (we describe this formally in Section 4). For consistency, for all rules, we assume that ties are broken in favor of candidates from party D but randomly within each party.[5]

2.2.2. Voter Assumptions. Next, we require assumptions on how voters rank or approve candidates. For each Thiele rule, we assume that, in each district k, there are exactly N_k candidates from each party, and each voter simply approves all the candidates in the voter's party.[6] For STV, we assume that, in each district k, there are at least N_k candidates from each party.[7] For our primarily empirical analysis, we further assume that parties are solid coalitions (Dummett 1984): that each voter ranks all candidates of the voter's party over each candidate of the other party though intraparty rankings may vary. These assumptions reflect that party membership is a strong predictor of the party of the candidate for which a voter votes. Such an assumption, using historical party vote shares to predict vote shares in future elections, is used throughout the redistricting literature for SMDs; our additional assumption is that this behavior extends to partisan voting in multimember elections. We relax this assumption in Section 4 and study intraparty effects when voters may disagree on within-party rankings in Online Appendix B.

2.2.3. Calculating Seat Share Given Vote Shares. Suppose—for a given district with N_k seats and V_k voters—we know the fraction y_p of voters that belong to each party $p \in \{R, D\}$ (vote share). How do we calculate the number of winners n_p from each party for a given social choice function?

Given our assumptions (including solid coalitions), it is straightforward to efficiently do so for Thiele rules as candidates from a given party receive the same votes, and so we do not have to consider individual candidates, only the number of winners from each party directly. Then, we have that the party R seat share is

$$n_R(y_R,\lambda) = \min_n \left[\arg\max_n \left[y_R \sum_{i=1}^{n} \lambda(i) + (1-y_R)\sum_{i=1}^{N_k-n} \lambda(i)\right]\right]. \quad (1)$$

The inner arg max comes directly from the rule definition; for a given n, suppose a fraction y_R approves n candidates. Then, that yields $y_R \sum_{i=1}^{n} \lambda(i)$ points with the remaining fraction $1-y_R$ approving $N_k - n$ candidates and yielding an additional $(1-y_R)\sum_{i=1}^{N_k-n} \lambda(i)$ points. The outer min simply encodes tie-breaking in favor of party D.[8] The party D seat share is $n_D(y_R,\lambda) \triangleq N_k - n_R(y_R,\lambda)$.

On the other hand, in STV, candidates of the same party may have different numbers of votes (in any round) as intraparty voter rankings may be arbitrary. One may, thus, naively, believe that calculating party seat shares requires further assumptions on voter behavior (on voters' intraparty rankings) and carrying out the iterative procedure defined above as the elimination and transfer scheme introduces substantial path dependence across candidates and rounds; in general, with arbitrary voter rankings, such calculations are required. Doing so with many voters and candidates would be prohibitively computationally expensive as a subroutine in a redistricting optimization algorithm. The following proposition establishes that, under our assumptions, party seat shares can be efficiently calculated (up to rounding) just from the vote shares without dependence on voter rankings.[9]

Proposition 1 (Seat Shares Under STV). *Suppose—for a given district with M seats and $V \ge M(M+1)$ voters—that a fraction y_p of voters belongs to each party $p \in \{R, D\}$ and that there are at least M candidates per party. Assume that each party's voters rank all same-party candidates above all other-party candidates, and ties are broken in party D's favor.*

Let $n_R(y_R, STV)$ be the number of winners belonging to party R using STV. Then, both $n_R(y_R, STV)$ and $n_R(y_R, \lambda_{\mathrm{PAV}})$ are in $\{\lfloor y_R M\rfloor, \lceil y_R M\rceil\}$.

At a high level, the proof establishes that no candidate receives meaningful (in terms of party vote share) votes from voters of the other party; in terms of party vote share, we can proceed with running two separate STV processes. Then, the sequential round path dependence does not matter as the overall number of first place votes for candidates of each party is invariant, and so the final partisan outcome depends just on initial vote shares. We note that the proof does not depend on our assumption of fractional STV; other transfer rules (such as random voter transfer) that preserve the number of surplus votes induce seat shares according to the same formula.

Importantly for this work, Proposition 1 enables calculating party seat shares in a district that uses STV without needing to (either computationally or in terms of data requirements) consider voters' individual rankings, substantially simplifying the task for a (human or algorithmic) map drawer.[10] Whereas simulating STV is not computationally intractable for a single election, in our computational optimization, we need to run STV elections in each district as a subroutine to map generation and optimization in the process of optimizing over about 10^{12} maps for a single state; as explained next in Section 2.3, we leverage this result to draw gerrymandered and fair maps for STV. Note that equivalence between PAV and STV is not true in general (Faliszewski et al. 2019). We use the relationship for the setting in this work to transfer intuition from PAV to STV.

2.2.4. Discussion of the Solid Coalitions Assumption.

A core rationale for ranked choice voting is that it allows elicitation of preferences beyond two-party identity; our primary methods, assuming solid coalitions, do not capture this rationale. However, we believe that the assumption is suitable for the goals of this paper to study partisan implications of MMDs in the United States. The United States has single-winner elections for the executive (president, governor) and so tends toward two parties for those offices. Whereas it is possible that the solid coalitions assumption substantially breaks for legislatures with MMDs, it is at least equally reasonable that the two-party structure approximately remains with subparties nested within them (preserving two-party solid coalitions). That said, formally extending results beyond solid coalitions is of substantial interest. The technical hurdle is finding an alternative assumption that (i) is believable for the political context and (ii) does not require simulating STV within the inner loop of a redistricting optimization, which is computationally prohibitive at scale. In other words, studying gerrymandering with social choice as we do requires some assumption to enable efficient calculation of seat shares in each district without simulation of elections. For example, more recent work by the MGGG Redistricting Lab (2022) uses the same assumption as we do when generating maps but then simulates ranked choice votes beyond solid coalitions (as we do in Online Appendix B) given those maps; their results, focusing on racial representation, are similar to ours.

We believe that finding a more reasonable assumption than (approximate) solid coalitions—that is still suitable for optimization—for the entire United States is unlikely, but such assumptions could be reasonable for specific political contexts (i.e., awareness of possible third parties or coalition splits that would form within a state). We leave such analysis for future work after more RCV elections have been run in the United States. In Section 4, we relax this assumption by calculating outcomes—on maps calculated by assuming solid coalitions—on simulated voter preferences when it does not hold; our insights generalize to this setting.

2.3. Empirical Method

We study our research questions empirically in the context of the U.S. House of Representatives. At a high level, we proceed as follows, separately for each state. Given historical voting data, we algorithmically generate maps for each state, social choice function F, and district size K: the most gerrymandered map for each party (the map that, given the voter data and the social choice rule, would produce the most winners from that party), the map with the smallest proportionality gap, and a neutral ensemble of random maps. For each map, we then calculate our metrics of interest for the relevant social choice rule. We detail steps of this process next. In Section 4 and Online Appendix B, we simulate full STV elections to study cross-party and intraparty effects, detailing the relevant methodological differences there.

2.3.1. Data. To calculate partisan seat shares, we require vote shares in each atomic block that are used to compose districts. For each multidistrict state, we use the geography-matched precinct-level statewide election returns from multiple election data repositories (Ansolabehere and Rodden 2011a, b, c, d, e; Massachusetts Institute of Technology Election Data and Science Lab 2018; Voting and Election Science Team 2018, 2019; MGGG 2020) for elections since 2008, aggregated by Gurnee and Shmoys (2021) (including presidential, senate, and statewide gubernatorial elections). These data include vote shares for each candidate in each precinct in a collection of state and national elections. We filter these results to just Republican and Democratic candidates and average the two-party vote share across elections at both the block level and the state level.[11] For the atomic geographic blocks, we use census tracts with population counts provided by the five-year American Community Survey estimates. As a general caveat, given the infrequency of statewide elections, these averages are noisy and sensitive to the idiosyncrasies of each race, candidate, and political environment as well as true underlying shifts over time.

For the analyses in Section 4 and Online Appendix B, we further require individual-level characteristics that can be used to simulate voter rankings. There, we use a voter file provided to us by a private election analytics company with individual-level voter data: in each block, an anonymized list of voters along with their (potentially modeled) demographics (including ethnicity, gender, census block of home address). Each voter is further assigned a party (either modeled or ground truth in states with party registration) and is scored on several ideological dimensions through a mixture of methods, including survey modeling, ecological inference, and individual-level voting history.[12] We calibrate the voter file to match the calculated party vote shares by subsampling the voters according to their assigned party; for replicability, we provide a subsample of this data set (with noise added) in our code repository.

2.3.2. Generating Maps. To generate maps, we extend the stochastic hierarchical partitioning (SHP) algorithm by Gurnee and Shmoys (2021), which recursively samples subdivisions of a state to create a large, random[13] ensemble of population-constrained and contiguous districts. These subdivisions are organized into a tree of nested regions that implicitly encode an exponential number of distinct maps. We extend the original tree growth logic to guarantee the existence of a split with child nodes each able to produce appropriately sized districts for a given MMD size policy. Each district (leaf node) is scored by expected party seat share for each social choice function,[14] and the tree can be efficiently traversed with a dynamic program to select the most gerrymandered map, or the tree leaves can be gathered into a secondary integer program to select the districts that create the most fair plan (for arbitrary definitions of fairness).

To ensure robustness, we estimate the variance of district vote share across elections to calculate the expected seat share per district and optimize for this expectation rather than simply optimizing for the number of strict majority seats. This prevents, for example, the algorithm from drawing districts in which the historical vote share was 50.1% for one party and declaring them safe seats for that party. We note that this also empirically lessens the sensitivity to the solid coalitions assumption if historical variance correlates with crossover votes in STV.[15]

All details of the original SHP algorithm can be found in the paper by Gurnee and Shmoys (2021) and associated appendices; we further use their same geographic and electoral data (see Gurnee and Shmoys 2021, appendix table 2). Here, we discuss the relevant algorithmic parameters and necessary modifications; see code for further details. The main algorithmic difference from the original SHP algorithm is that we adapt ours to generate multimember districts. To do this, instead of parameterizing a sample tree node by a region R and total number of seats s, we needed to also specify the number of districts that node contains. This is required because, at an intermediate node, the number of districts is not immediately derivable from the total number of seats in that node because of ambiguity in the number of N_k versus $N_k + 1$ sized districts (and, analogously, in our experiments simulating three-, four-, or five-member districts in a state as in the Fair Representation Act). Therefore, the number of seats is used just to balance the population, and the number of districts is used for all other tree operations (sampling valid splits, maintaining balance, etc.).

For each state, let N be the number of seats for that state in the House of Representatives. Then, we construct maps for that state with K districts for each of $K \le N$ and within $\{2,3,\ldots 10\} \cup \{12,14,16,18,20\} \cup \{23,26,29,\ldots 53\}$, where 53 is the largest number of seats for any state.[16] For each pair (state, K), we sampled the root node $\left(\frac{1{,}000}{K}\right)^{1.2}$ times and each internal node $\left(\frac{300}{K}\right)^{0.5}$ times. These constants were chosen to balance computational cost and optimization quality. We used random-iterative center selection with Voronoi-weighted capacity matching to sample region centers and sizes. All districts are population-balanced within a 1% tolerance as described in Section 2.3. Each of these ensembles was then scored, optimized, and subsampled to create a final distribution over partisan outcomes. For the subsample used to calculate the distribution of outcomes under neutral maps, we use $1{,}500\sqrt{K}$ maps for each (state, K) pair. The number over which the optimal maps (most fair, most gerrymandered) are calculated is more complicated: these maps are implicit in the tree structure. The set of tree leaves (districts) can be used to compose an exponentially large number of maps; we can optimize over all these maps without enumerating them through a dynamic program (for most gerrymandered maps) or integer program (for most fair maps). For each setting, there are between 10^3 (smallest states) and 10^{12} (largest states, most districts) such implicit maps that are optimized over.

2.3.3. Comparison with Other Methods.

We adopt this approach rather than the standard recombination Markov chain algorithm (DeFord et al. 2021) and related local update methods (Autry et al. 2020, 2021; Fifield et al. 2020; Cannon et al. 2023; McCartan and Imai 2023) to more efficiently model both partisan and nonpartisan mapmakers under multiple social choice functions. Our approach generates just one district ensemble (set of districts as leaf nodes of the tree) that can then be combined into one of exponentially many valid maps (in number of districts), optimizing each objective under each modeling scenario; in contrast, a single neutral recombination map ensemble is unlikely to capture the full range of potential partisan gerrymandering in every setting, necessitating the use of a separate biased chain optimized for each combination of party and voting rule and increasing the amount of computation by almost an order of magnitude. Furthermore, in a policy setting with a variable number of districts (as in the Fair Representation Act), a single hierarchical sample tree produced by SHP can efficiently capture all possible combinations of district sizes. In contrast, recombination would either require running 29 separate chains for the 29 different combinations of three-, four-, and five-member districts to fill California's 53 seats under the Fair Representation Act rules (times each combination of party and social choice rule for the full experiment) or would require creating rules for how to combine and repartition districts of varying sizes. For example, when studying the effects of the Fair Representation Act, MGGG Redistricting Lab (2022) considers just one combination of district sizes per state.

On the other hand, an advantage of recombination methods is that they sometimes come with guarantees of the sampling distribution over maps, whereas no such analysis is available for the SHP algorithm. However, in the context of SMDs, Gurnee and Shmoys (2021) find that SHP finds more extreme maps than other methods do.

2.3.4. Our Experiments.

Putting things together, for each generated map, we can simply calculate our metrics of interest. We carry out the process for each social choice function and each state with two or more seats in the House, fixing as N the number of seats allocated to that state after the 2010 census. The number of districts K is from one (a large MMD with N seats) to N (only SMDs). For each K, the set of district sizes $\{N_k\}$ is selected such that all districts differ in size by at most one seat[17] except in the analysis of the Fair Representation Act, in which case we generate maps with districts of size either three, four, or five.

Overall, generating districts and calculating the metrics utilized over 40 CPU-weeks of computation, not counting the full STV elections simulated in Online Appendix B.

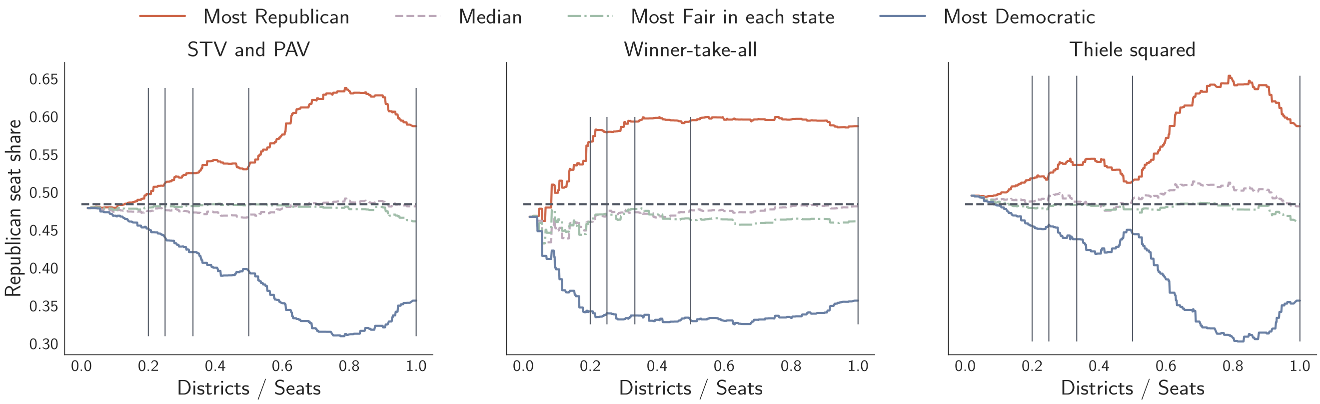
3. Results: Interparty Balance with Multimember Districts

We first analyze how MMDs affect partisan balance in both how MMDs empower independent commissions and how they constrain adversarial partisan gerrymanderers. Section 3.1 contains our main results; we illustrate the effects of each social choice function and district size for both extreme gerrymanders and fair redistricting. In Section 3.2, we turn to design questions, discussing the effect of the Thiele rule parameter and heterogeneity across states in terms of optimal district size; here, we further analyze how the Fair Representation Act navigates such trade-offs.

3.1. Gerrymandering, Proportionality, and Competitiveness with Multimember Districts

Figure 1 contains our main result; for each social choice function, it shows how the overall (across states)

Figure 1. (Color online) The Republican Seat Share over All States as the Number of Districts Is Varied in Each State



Notes. The horizontal line denotes the vote share fraction, that is, the proportionality value. Each point aggregates over every state with rounding and weighting by number of seats in the state and vertical lines corresponding to when N/K, the average number of seats per district, is an integer. For example, the vertical line at 0.5 corresponds to two-member districts in states with an even number of seats and all two-member districts except one one-member or three-member district in states with an odd number of seats. The rightmost point is with SMDs, and the leftmost point is if each state has one large MMD. "Median" refers to the median value found across random maps from the SHP algorithm and "Most fair in each state" to the maps with the smallest proportionality gap. Overall, MMDs are effective at preventing the worst gerrymanders, especially with non–winner-take-all rules. Note that national seat share with median and fair maps only look invariant to district size because gaps cancel out between states (some favor Democrats, others Republicans). Figure 2 shows that seat share is not invariant at the per-state level: even when optimizing for fairness, we require MMDs to reduce the per-state proportionality gap.

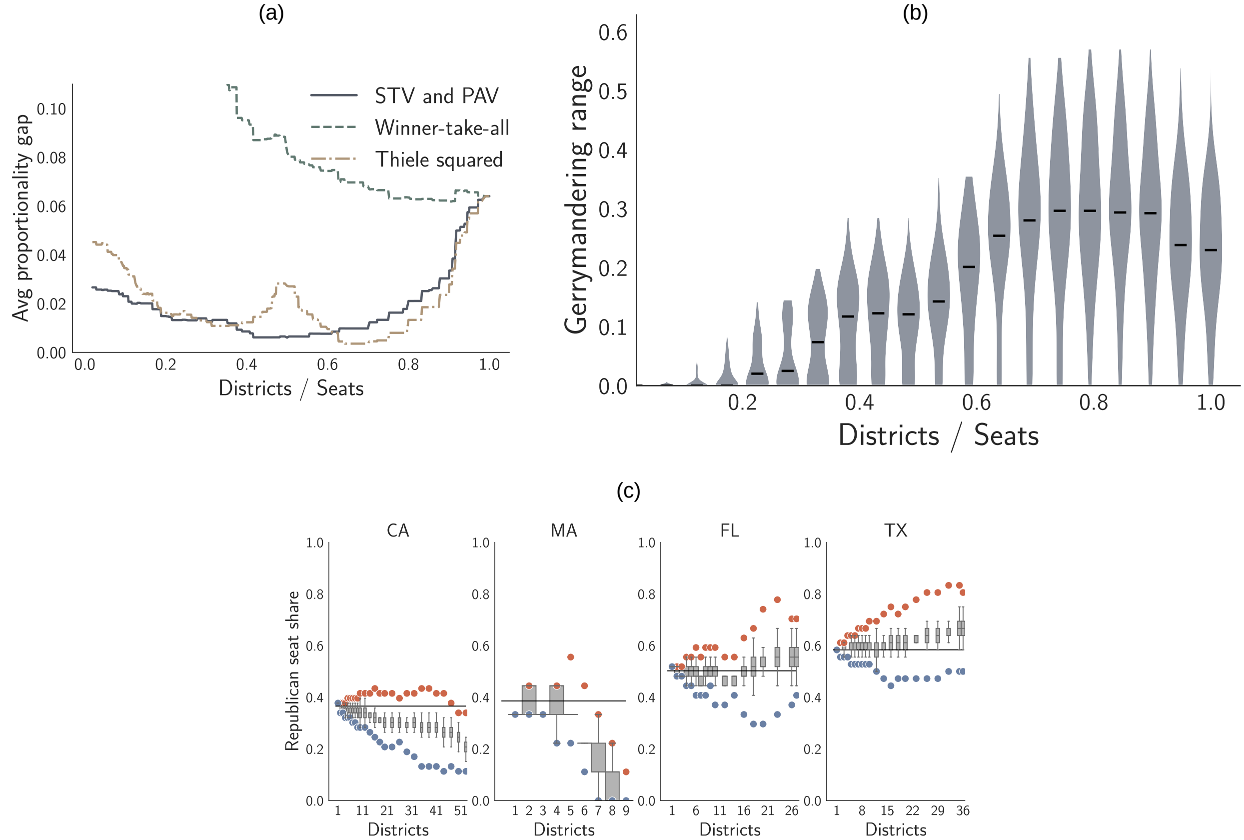
partisan seat share varies with the number of districts. Figure 2 zooms in on the state level.

3.1.1. Preventing Intentional Gerrymanders. MMDs are effective at preventing the most extreme gerrymanders but only with non–winner-take-all rules. As expected, in the limit, with one large MMD in each state and using STV (equivalently, PAV), the proportionality gap in each state is negligible. Perhaps more surprisingly, just moving from SMDs to two-member districts provides about half the benefit in terms of reducing the proportionality gap for the most extreme gerrymanders. Three-member districts further reduce this gap, for example, from a maximal seat share of around 60% to around 57% for Democrats. With five-member districts, the range of outcomes is between 45% and 50% Republican seat share even if one party controlled all redistricting nationally. To further understand the role of gerrymandering as district size increases, define the gerrymandering range as the difference in Republican seat share between extreme Republican-optimized and Democrat-optimized maps; a small range suggests that outcomes would not vary much within the range of feasible maps. Figure 2(b) shows, for each district size, the distribution of the gerrymandering range across states. Whereas the range varies substantially at the state level for SMDs, it shrinks across states as district size increases, showing that the effect appears at both the state and national level.

What explains these results? With SMDs, an R gerrymander would draw a map such that as many districts as possible have a majority of Rs (up to a tolerance for robustness). It does so by cracking, creating many districts in which the D vote share is just below $\frac{1}{2}$ (thus electing an R candidate), and, as necessary, packing, creating a few districts in which the D vote share is as close to one as possible. Such a map maximizes D wasted votes. Now, consider PAV, that is, a Thiele rule with $\lambda(i) = \frac{1}{i}$, and two-member districts. Any district in which the D vote share is between $\frac{1}{3}$ and $\frac{2}{3}$ would elect one member from each party. Cracking, thus, requires creating districts in which the D vote share is less than $\frac{1}{3}$. Packing—wasting D votes—could mean targeting the D vote share to be just below $\frac{2}{3}$ (e.g., a D vote share of 0.6 to elect one member for each party) or close to one but then giving up both seats in the district. In each case, gerrymandering requires more precision and wastes fewer votes for the opposing party.[18] Simultaneously, having fewer districts to draw reduces the degrees of freedom available. The corresponding bands become narrower as district size increases.

The trend toward proportionality is not monotonic; a mixture of district sizes increases degrees of freedom, enabling gerrymanders even more extreme than those possible under SMDs. Consider one- and two-member districts; the gerrymandering party R could then pack or crack party D in the one-member districts and use the two-member districts to elect one candidate from each party with R vote share just above $\frac{1}{3}$. Similarly, an urban gerrymandering party D could waste fewer votes winning one-member urban districts, still getting above $\frac{1}{3}$ of the vote in two-member rural districts. (In other words, all but the most extreme two-member districts will result in one member elected from each party

Figure 2. (Color online) How the Partisan Lean and Proportionality Gap Vary at the State Level with Voting Method and the Number of Districts



Notes. (a) The absolute value of the state-wise proportionality gap in the "most fair" map in each state. Even if a redistricting agent wanted to close the proportionality gap, it could not do so with SMDs. (b) The distribution of the "gerrymandering range" (difference in Republican seat share between extreme Republican-optimized and Democrat-optimized maps) as a measure of how much redistricting matters. Each violin plot shows the distribution over states. Whereas the range varies substantially at the state level for SMDs, it shrinks across states as district size increases. (c) The per-state version of Figure 1, showing the full distribution of maps and the extreme gerrymanders for four states. Whereas there are substantive state-level gaps with SMDs even in the most proportional maps, the gaps become negligible with even just two-member districts and STV. Note that, in Figure 1, these gaps cancel out nationally as some state-wise gaps favor Democrats and others favor Republicans. Qualitatively similar results to panel (a) are shown for the median and gerrymandered maps in Online Figure 8, and plots for all states as in panel (c) are in Online Figure 6. (a) Average per-state magnitude of gap in "most fair" maps. (b) Gerrymandering range under STV and PAV. (c) Partisan lean distribution under STV and PAV.

with STV.) The pattern repeats, to a lesser extent, between every integer division.

Similar arguments apply to other Thiele rules with strictly decreasing λ and are amplified with sharper diminishing returns. Consider the Thiele squared voting rule. It favors—even more than PAV—50/50 outcomes in each district because voters who have fewer of their preferred candidates elected gain more from having an additional candidate elected than voters who already have many of their preferred candidates elected. Thus, when there are two seats per district, the Thiele squared rule strongly incentivizes outcomes in which each party wins one seat in each district. In Figure 2, this increases the proportionality gap for Thiele squared voting rules when there are about two seats in each district as 50/50 may not be proportional.

3.1.2. Enabling Proportional Redistricting. The above analysis considers the most extreme gerrymanders. However, commissions that draw fair maps are increasingly common (Cain 2011, Spencer 2020); we now show that MMDs enable the drawing of proportional maps that would not be possible under SMDs. Consider Figure 2(a), which shows the average absolute proportionality gap in each state for the map that minimizes this gap. We find that there is a substantive state-level gap that virtually disappears—up to rounding with a fixed number of districts—even with two-member districts and STV.

Figure 2(c) further demonstrates the SMD Massachusetts problem as elucidated by Duchin et al. (2019) and shows that MMDs fix it.[19] Because of how evenly Republicans are distributed (diffused) across the state, there is no way to draw SMDs such that a proportional number of Representatives are Republican. Intuitively, suppose party R has vote share $\frac{1}{3}$. With SMDs, to achieve proportionality, a commission would need to draw districts such that party R is in the majority in $\frac{1}{3}$ of the districts, which may not be possible or may require atypical, contorted maps.

A single MMD with PAV provably approximately achieves proportionality (Skowron 2021), solving the Massachusetts problem up to rounding. Our results indicate that even multiple two-member districts enable fair maps and solve the problem in practice. Intuitively, in the above example, a commission would just need to draw a map with enough districts with R vote share above $\frac{1}{3}$. Note that, by the pigeonhole principle, with overall party p vote share v_p, for any map, there will always be at least one district in which its vote share is at least v_p; so, as the threshold to win one seat decreases with district size, an ever-smaller minority party is guaranteed at least one seat.

Finally, we note that, as illustrated by the average proportionality gaps of the median maps in Online Figure 8, even small multimember districts with PAV or STV eliminate the issue of natural gerrymanders, in which even neutral maps—drawn by a redistricting algorithm that ignores partisan vote shares—favor one party because of the natural geographic distribution of voters. Such natural gerrymanders play a central role in discussion of a justiciable gerrymandering standard; our results indicate that, with even small MMDs, an independent commission would not need to draw maps that substantially differ from neutral maps in order to achieve proportionality.

3.1.3. Competitiveness. Up to now, we have primarily considered proportionality: how the partisan seat share reflects the underlying vote share in each state. Here, we consider competitiveness: the average vote shift needed to shift the number of seats won by each party by at least one. For example, in a winner-take-all election, a Republican vote share of 0.6 in a district would lead to a margin of victory of 0.1. In a PAV election with two winners, the same district would have a victory margin of $(\frac{2}{3} - .6)$ as a Republican vote of $\frac{2}{3}$ (with appropriate tie-breaking) would lead to two elected Republicans as opposed to one. Whereas competitiveness is controversial as a goal for redistricting (as an objective when drawing maps) (DeFord et al. 2020), it is considered an important dimension along which to evaluate a map. Some claim that uncompetitive districts, for example, could depress participation or contribute to polarization (Altman and McDonald 2015, Cancela and Geys 2016, Moskowitz and Schneer 2019, Gerber et al. 2020, Ainsworth et al. 2024).

One potential concern with multimember districts is that they may lead to uncompetitive districts if the number of members in each district is small. With two-member districts, for example, most districts will have one member from each party, and one of the parties would need at least $\frac{2}{3}$ of the vote to win both seats in the district. Our results, however, indicate that this concern is overstated. Figure 3 illustrates the average vote shift needed to shift the seat share; regardless of which map is considered, with PAV or STV competitiveness increases with district size (the average vote shift necessary to change at least one seat goes down). However, it is the case that most two-member districts would be split 1D-1R even in what would have previously been considered politically safe regions. This effect is amplified with Thiele squared voting rules, which further favor balanced outcomes.

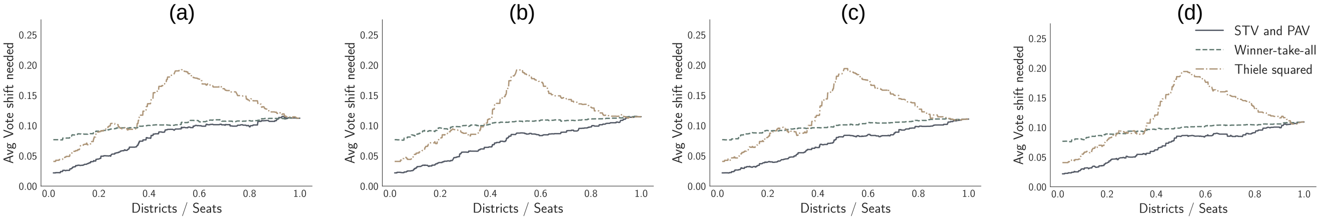
3.2. Design Recommendations and Discussion

In Section 3.1, we primarily discuss how partisan balance—proportionality and competitiveness—vary with respect to district size, generally across settings. Here, we expand on the role of the exact social choice function as well as heterogeneity across settings when it comes to the effect of district size. We note that our results—especially the nonmonotonicity, heterogeneity, and nonasymptotic behavior in district size—underscore the importance of an empirical approach to supplement theoretical analysis.

3.2.1. Effect of Social Choice Function. Perhaps the most obvious takeaway is that, using a winner-take-all procedure with MMDs, as was popular in the 20th century and still in effect in some state legislatures, enables rampant gerrymandering at nearly all district sizes and should be avoided (in fact, whereas proportionality gaps with one large winner-take-all MMD in each state cancel out at the national level in Figure 1, as shown in Figure 2(a), they increase to close to the theoretical maximum with larger MMDs).

Our results also illustrate the differential effects of the Thiele design parameter λ, especially as a function of district size. Our experiments test $\lambda(\cdot) = 1$ (winner take all), $\lambda(i) = \frac{1}{i}$ (PAV), and $\lambda(i) = \frac{1}{i^2}$ (Thiele squared). Among these, we find that, as we increase the rate at which λ decays in i, the corresponding Thiele rule more strongly results in legislatures that are equally composed of members of different groups regardless of the size of the underlying group. Especially with even-member districts, then, such a rule leads to proportional representation when the parties are approximately equal-sized—even more so than does STV or PAV—at the cost of proportionality for settings with imbalanced party size. For example, consider the difference between the national-scale proportionality of Thiele squared in Figure 1 with the per-state proportionality gap in Figure 2 with two-member districts. More generally, our empirical framework may be

Figure 3. (Color online) The Average Vote Shift Needed in Each Map (Averaged Across Districts in a Map and Across States) to Shift the Number of Seats Won by Each Party by at Least One



Notes. The larger the vote shift needed in a district, the less competitive the district is. The four plots show the various maps selected. For example, panel (a) shows the vote shift needed when the median (in terms of party seat share) maps are selected. With PAV/STV, competitiveness generally increases with district size. (a) Median map. (b) Most fair map. (c) Most republican. (d) Most democratic.

helpful in complementing theoretical analysis of Thiele rules (see, e.g., Skowron 2021) in understanding the effect of the design parameter in a given real-world setting given the actual distribution of voters.

3.2.2. District Size Design Heterogeneity and Recommendations. Our discussion above suggests that two- or three-member districts (as possible depending on the number of seats) with STV or PAV would work well across most political conditions in the United States. In particular, fairness-minded independent commissions can achieve proportional outcomes in every state up to rounding, and advantage-minded partisans have their power to gerrymander significantly curtailed.

However, there is also important heterogeneity; individual states can further promote proportional outcomes by tuning MMD parameters based on their population, partisan lean, and political geography. See Online Figure 6 for full state-by-state results; here, we discuss the most relevant factors influencing optimal district size. We observe that small and highly partisan states both benefit from larger districts. Smaller states have fewer opportunities for wasted votes to cancel out across districts and so can more easily become disproportionate. In contrast, highly partisan states often suffer from the Massachusetts problem, in which minority parties have difficulty breaking the requisite threshold in any district because of their diffuse voter base. In general, states with significant geographic self-segregation could protect the political power of both concentrated and diffuse voters with larger districts that inherently waste fewer surplus and losing votes.

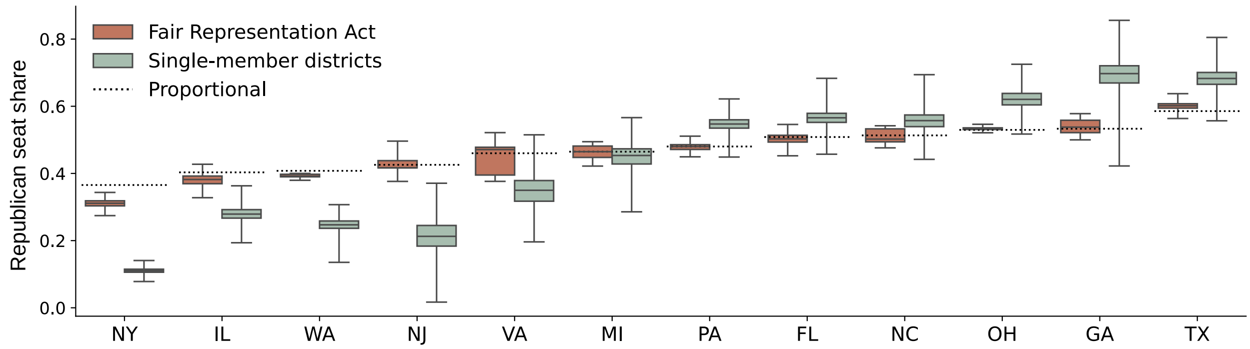
As such, flexibility across states may be beneficial; however, as Figure 1 shows, such flexibility may also increase gerrymandering if, within a state, partisan gerrymanderers can use a mixture of district sizes. We evaluate this trade-off by studying the potential outcomes of redistricting under the Fair Representation Act, a proposal mandating that the number of seats per district satisfies $N_k \in \{3, 4, 5\}$. As before, we sample random maps and construct optimal maps for each party when a map can have districts of size three, four, or five such that the total number of seats adds up to the number of seats in the state. Figure 4 suggests that this choice would allow flexibility without giving gerrymanderers too many degrees of freedom in district size: it enables proportionality almost everywhere, and it substantially reduces gerrymandering capacity over the baseline of using just single-member districts. Online Figure 10 further shows that such flexibility does not offer significant additional capacity to gerrymander over a baseline of using just three- or just five-member districts in contrast to the case of mixing single- and two-member districts.

This finding is important because coherent communities substantially vary in size and so are best represented by variable-sized districts. For instance, in Texas, the Houston metro area may be best covered by a five-member district, whereas San Antonio, a city with roughly half the population, may best be covered by a three-member district. Under a fixed number of seats per district, either Houston is split or San Antonio is grouped together with disjoint communities. The Fair Representation Act's approach would allow both district sizes. In Online Appendix B, we continue evaluating such trade-offs, studying intraparty effects of MMDs with STV, for which the solid coalitions assumption no longer holds.

4. Results: Beyond Solid Coalitions

So far, we have made a solid coalitions assumption with STV: that each voter ranks all candidates of the voter's own party above all candidates of the other party. In practice, voters may not behave this way and may cross party lines in how they rank candidates in any given election. In this section, we empirically show that such crossover voting does not affect our primary results—that multimember districts with STV blunt the effect of gerrymandering—under a standard model of such voting. Intuitively, gerrymandering with any number of districts becomes less effective with unpredictable cross-party voting (because the optimization target has higher variance), and we empirically find

Figure 4. (Color online) Comparison of Republican Seat Share Distribution of the Fair Representation Act–Compliant Ensembles Using the STV/PAV Voting Rule to Single-Member District Baseline for a Selection of States



Note. See Online Figure 9 for additional voting rules.

that this effect is additive to the effects of multimember districts.

The biggest challenge to studying such effects is that Proposition 1 does not hold without the solid coalitions assumption, and so we must now generate voter rankings and simulate STV for each district in each potential map.[20] To simulate voter rankings, we leverage additional data that characterizes individual voter partisan lean and then construct voter rankings over simulated candidates using a random utility model (RUM).

4.1. Methods and Assumptions

To study settings beyond the solid coalitions assumption, we first construct plausible voter rankings and candidate distributions; second, we simulate STV elections given a map.

4.1.1. Voter Generation and Multidimensional Preferences. The key step is to develop a model for how a voter will vote given a menu of (hypothetical) candidates. Up to now, we have only assumed that voters approve all candidates of their party or rank them all above candidates of the other party, but here, we require a model for how voters rank candidates both within and across parties. We do so as follows, using the voter file described in Section 2.3. Each voter in our data set is assigned scores along several ideological and demographic dimensions derived from survey responses and administrative records. In this work, we use a single scalar partisan score, taking values in $[0, 100]$, which represents the modeled probability that the voter supports the Democratic candidate in a two-party general election. This score is constructed using party registration, historical primary participation, demographic variables, and related covariates and is shown to correlate strongly with issue-specific preference scores (PredictWise 2021). We threshold the partisan score at a fixed value of 50: voters with scores above the threshold are classified as Democrats and those below as Republicans. Each voter is also associated with a census tract corresponding to the voter's home location. Given a map, this uniquely assigns each voter to a district. The data set is rebalanced to match the state-level vote shares in recent statewide elections as computed in Section 2.3. For computational tractability, in each simulation, we sample at most 50,000 voters per state. For replication purposes, preserving data privacy, in our code repository, we provide a subsample of 50,000 voters with noise added to the scores.

4.1.2. Candidate Generation. Candidates are generated independently for each (party, census tract) pair. For each such pair, we create a single candidate whose partisan score is set to the median partisan score among voters of that party residing in the tract. This candidate is eligible to run in any district containing that census tract. For computational reasons, we cap the total number of candidates per state at 1,000, sampling at most 500 candidates from each party.

4.1.3. Voters and Candidates in an Election. A map partitions census tracts into districts. Because both voters and candidates are associated with census tracts, each belongs to exactly one district under a given map. In a given district-level election, voters may rank only candidates whose census tracts lie within the same district. Depending on the map, a district may contain a large number of candidates (for example, if the district spans most or all of a state). For both computational cost and statistical reasons arising from large candidate pools, we further subsample candidates within each district.

Specifically, from each party, we sample at most a number of candidates equal to the number of seats to be filled in the district. This restriction is imposed for the following reason. Under the random utility models we consider, as described below, utilities are composed of a deterministic component that favors ideologically proximate candidates and an independent and identically distributed (i.i.d.) noise term. When the candidate pool is very large, extreme-value effects dominate: with max concentrating noise distributions such as the normal distribution, the highest utility candidates for a given voter are all drawn from the voter's own party with high probability. As we aim to study crossover voting and go beyond the solid coalitions assumption, we subsample the candidates. This subsampling can reflect a primary or a natural limitation on the number of competitive candidates from each party in a district, enabling meaningful crossover votes.

4.1.4. Turning Voter Preferences into Rankings. In each simulated election, we need a ranking over candidates from each voter. We generate complete rankings over candidates using an RUM. Let s_v denote the partisan score of voter v and s_c the partisan score of candidate c. The utility that voter v assigns to candidate c is

$$u(v,c) = -|s_v - s_c| + \varepsilon_{vc},$$

where ε_{vc} are i.i.d. idiosyncratic values drawn from a distribution F. We consider two choices for F: (i) a normal distribution with mean zero and standard deviation σ and (ii) a Gumbel distribution with location parameter zero and scale parameter β (implying mean 0.5772β and variance $\frac{\pi^2}{6}\beta^2$). We report results for $\sigma, \beta \in \{0, 5, 50, 100, 250, 500\}$. When ε_{vc} follows a Gumbel distribution, the induced distribution over rankings coincides with the Plackett–Luce model (Luce 1959, McFadden 1972, Plackett 1975). When ε_{vc} is normally distributed, this corresponds to the classical Thurstone–Mosteller random utility model (Thurstone 1927, Mosteller 1951). Unlike the Gumbel distribution case, normal noise does not satisfy independence of irrelevant alternatives. Whereas both models differ in their tail behavior, we find that our qualitative insights do not change with the noise model considered. For intuition on the noise levels with a Gumbel random variable with $\beta = 50$, in our data, a voter ranks a candidate from the other party first with probability about 25%; with $\beta = 5$, this probability is about 5%. However, we note that the mapping from such crossover probabilities to map robustness is not straightforward given that we subsample candidates and voters. Thus, we report results across a range of noise levels to show that our insights are robust across different levels of crossover voting.

Finally, unlike in actual elections, we assume that there is no ballot exhaustion. Each voter submits a complete ranking of all candidates in their district, ordered by decreasing realized utility.

4.1.5. Simulating STV Elections. Given the candidates, sampled voters, and the voters' rankings over the candidates, we run fractional STV for each district in each given map to determine the elected candidates. Fractional STV is STV as defined above. In each round, either a winner is selected if the candidate has at least the droop quota number of first place votes, or a candidate with the least votes is eliminated. This candidate's votes are transferred by eliminating this candidate from each voter's list. In fractional STV, votes are transferred as follows. Formally, suppose the winning candidate receives $v > Q$ first place votes. Then, a fraction $\frac{v-Q}{v}$ of this candidate's votes is in excess of what is needed to win. Thus, each voter for this candidate has the voter's weight multiplied by $\frac{v-Q}{v}$. For example, suppose the droop quota was five, and a candidate received 6.3 first place votes (fractional votes arise from earlier round transfers). Then, each of the voters has the voter's weight multiplied by $\frac{1.3}{6.3}$ for the next round.

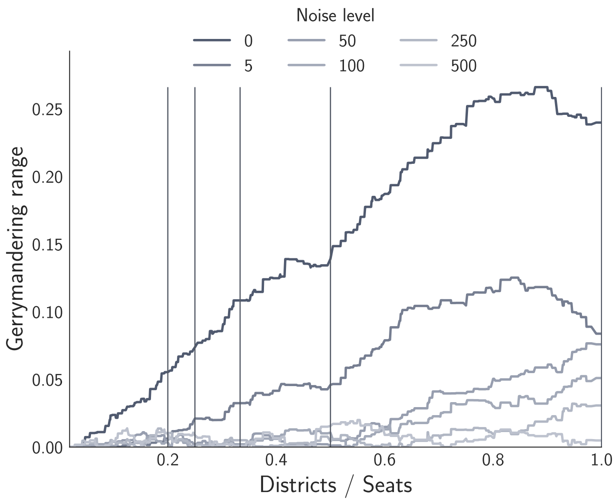
Running these STV elections utilized over 120 CPU-weeks of compute on top of the map generation discussed in Section 2.3. This runtime is for given maps, underscoring the necessity of Proposition 1 to study partisan seat shares without needing to simulate STV during the redistricting optimization process.

4.1.6. Discussion. There has been much work on spatial voter models aiming to characterize such behavior based on the distance between the voter and each candidate (Jessee 2009, Adams et al. 2017, Shor and Rogowski 2018, Tausanovitch and Warshaw 2018). Random utility models in general and the two we use in particular reflect spatial theories of voting (Enelow and Hinich 1984, Merrill and Grofman 1999, Azari et al. 2012) and are widely used in the social choice and political science literatures to simulate preference profiles and analyze voting rules. However, modeling voters' ranked preferences is a hard challenge; it is not clear that voters behave according to such ideological spatial positioning. Thus, our results should not be interpreted as what would necessarily happen with multimember districts. We note, however, that we expect our main insight—that multimember districts blunt the effects of gerrymandering—to hold more generally as gerrymandering becomes less effective when voter behavior is less predictable. Our approach here provides a first step toward quantifying such effects by providing a plausible model for voter behavior and simulating STV elections accordingly.

4.2. Results

Figure 5 contains our results with Gumbel noise, showing how the gerrymandering range contracts both with increased voter noise (i.e., crossover voting) and district size with the effects being cumulative. In particular, the zero-noise line shows the gerrymandering range (difference in Republican seat share between Republican-optimized and Democrat-optimized maps) when votes are simulated using the above approach (with sampled voters and synthetic candidates); we average across the 25 most extreme maps calculated for each party for

Figure 5. (Color online) Gerrymandering Range (Difference in Republican Seat Share Between Republican-Optimized and Democrat-Optimized Maps) for Different Voter Ranking Noise Levels to Simulate Cross-Party Voting



Notes. Here, the idiosyncratic noise term for each voter–candidate pair is drawn from a Gumbel distribution with the noise level corresponding to β (with higher noise corresponding to higher variance). Results with normal distribution noise are in Online Figure 13 and are qualitatively similar. As noise levels increase, the range of seat shares feasible using gerrymandering decreases, and the effect is in addition to that of using bigger districts with STV. Intuitively, higher levels of voter noise correspond to gerrymandering being harder because individual voters differ more from what is predicted.

each setting with zero noise. Results without noise largely reflect the range in Figure 1 with some noise introduced by our voter sampling process (recall that we rebalanced our voter file to reflect the state-wise seat shares but not necessarily at each precinct).

Then, for each of the other lines, we show the empirical gerrymandering range for the same set of maps as optimized in the zero-noise case; now, however, voter rankings over candidates are noisy as described above. This reflects a setting in which parties would gerrymander without factoring in crossover voting (in practice, they might factor in the potential of crossover voting but not each voter's realization).

As voter noise increases, there are more crossover votes, and the gerrymandering range decreases. At a high level, the result establishes that crossover voting does not affect the qualitative insights of our work: the gerrymandering range decreases with STV as districts become larger, and much of the benefit is realized with three-member districts. The effect of noise is intuitive as it makes the optimizer's task more difficult when it is not foreseeable (the optimizer draws maps assuming zero noise). Online Figure 13 replicates this plot using normal noise, and Online Figure 14 shows the result when the noise is perfectly foreseeable, that is, the optimizer observes the voter ranking realizations over candidates when drawing maps, and different maps may lead to different candidate and ranking realizations. Both alternative analyses replicate the results; in the latter case with foreseeable voter rankings, noise increases gerrymandering ability (because the additional noise yields more extreme voter preferences), but gerrymandering range decreases with district size.

More broadly, this section gives an approach to evaluating voting rules and redistricting under more complex voting preferences: generate voter rankings from individual-level data, simulate the social choice rule on a fixed set of maps, and compare outcomes along interpretable dimensions. Here, we relax solid coalitions across parties, allowing voters to rank some opposite-party candidates highly through a random utility model. Online Appendix B uses the same high-level approach but focuses on intraparty effects. As a result, it retains party-line ordering across parties but allows voters within the same party to disagree about same-party candidates. There, voters rank candidates according either to partisan-score proximity or geographic proximity (and without noise), allowing us to study how MMDs affect intraparty diversity and geographic cohesion. The appendix results suggest a trade-off: larger MMDs can increase within-party diversity among winners, but very large MMDs weaken the geographic cohesion of the voter coalitions supporting each representative. Thus, medium-sized MMDs may be a sweet spot for balancing these effects, also achieving most of the proportionality benefits of larger MMDs (as shown in Section 3).

5. Discussion

5.1. Discussion and Limitations

In this work, we focus empirically on the United States and on MMDs. There are many other systems designed for proportionality in use internationally; a comparative study would be of use. Future work can also adapt our methodology to study partisan gerrymandering in state and local legislatures, especially in the states that already use (winner-take-all) MMDs. Such legislatures stand to benefit even more from MMDs than congressional districts given that they cannot rely on interstate cancellation of partisan bias as is currently the case in the House of Representatives. Whereas we expect the high-level insights to hold, details may differ when zooming in with more seats in a smaller region because of, for example, effects of geography.

There is much left to do to understand multimember districts in practice, especially how they relate to other constraints and laws. (i) The 2021 version of the Fair Representation Act has a provision[21] that MMDs should not be used in states where doing so would violate the Voting Rights Act; it would be crucial to analyze under what circumstances this could happen and how one would measure violations, for example, by diluting the voting power of racial minorities. (ii) More generally, we impose no constraints beyond contiguity and population balance, but many areas impose additional requirements (such as minimizing the splitting of county boundaries). (iii) We assume that voters rank all candidates when, in practice, there may be ballot exhaustion. We note that our main results show the effectiveness of two- and three-member districts, in which voters may be more likely to submit full rankings. Cataloging such constraints and enforcing them computationally is challenging; we conjecture that adding such features from practice is important but will have second order effects. Also crucial is studying how intraparty effects such as those studied in Online Appendix B (and in the work of, e.g., Angulu et al. (2019) and Buck et al. (2019)) interact with partisan gerrymandering (or gerrymandering on other characteristics).

5.2. Conclusion

We study the joint gerrymandering and social choice problem, showing the promise of multimember districts with non–winner-take-all rules in ensuring partisan proportionality in terms of both enabling independent commissions and constraining partisan gerrymanders. As an application of our approach, we computationally analyze the Fair Representation Act with three- to five-member districts in the United States. Finally, whereas this work is empirical, it raises theoretical questions that we expect to be of interest to the computational social choice community. For example, much is known theoretically about proportionality in the case of one large MMD (see, e.g., Skowron 2021 and references therein). Theoretically analyzing proportionality in the case of multiple MMDs, under random and adversarial ways to construct districts, would be of interest. Our results empirically show that under party-based voting, the proportionality gap decreases (nonmonotonically) with district size; proving that this effect theoretically under more general assumptions, such as those empirically shown in Section 4 with a noisy voter ranking model, is of interest.

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Endnotes

[1] See https://beyer.house.gov/news/documentsingle.aspx?DocumentID=5276.

[2] In this work, we use "fair" and "proportional" interchangeably. There are many other redistricting desiderata in the literature, which we briefly discuss in the related work and also show results for in Section 3. Similarly, we refer to independent commissions as one group that may desire proportionality though, in practice, such commissions have other possible objectives.

[3] Our chosen tolerance provides a 1% upper bound in population deviation. More precisely, in the optimization, let $\tau = 0.01$ and the single-seat ideal population be $P_{\text{ideal}} = P_{\text{total}}/N$. For a district with N_k seats at tree depth L, we enforce $P_k \in [N_k P_{\text{ideal}} \pm (\tau P_{\text{ideal}}/L)]$, that is, a relative deviation $\delta_k = \tau/(LN_k)$ from $P_{\text{ideal},k} = N_k P_{\text{ideal}}$, bounded above by 1%. This modeling tolerance should not be interpreted as a legal threshold: for U.S. congressional redistricting, the constitutional standard is population equality "as mathematically equal as reasonably possible." The U.S. Supreme Court has struck down a plan with less than a 1% deviation between the smallest and largest districts (American Civil Liberties Union 2001).

[4] Of course, to the extent that these laws are enforced in court, maps have to be in accordance with the Voting Rights Act and any state-specific laws concerning partisan balance.

[5] Tie-breaking is assumed for theoretical precision. It does not play a role in our empirical analysis given the data.

[6] For example, with SMDs, each party typically advances exactly one candidate chosen via a primary.

[7] For approval voting rules (such as the Thiele rules), having more candidates from each party than the number for which a voter is allowed to vote would induce stronger strategic concerns and might not lead to proportionality even with solid coalitions. For example, suppose there are two candidates from each party in a district with one seat; then, because each voter can vote for only one candidate, the party with a majority of the votes might not win the seat if its voters split their votes between the two candidates; for this reason, in standard elections with one winner and one vote per voter, there is a primary for each party to decide its candidate. For ranked voting rules, such as STV, in which the voter can rank all candidates, having more candidates than seats does not create such concerns because voters can rank all candidates in their preferred order.

[8] Ties occur when $y_R\lambda(i+1) = (1-y_R)\lambda(N_k - i)$ for some i.

[9] The first part of the result is a known property of STV (see, e.g., Dummett 1984, Tideman 1995) under an assumption of solid coalitions as we have here. The second part also follows from, for our setting, the equivalence between PAV, d'Hondt's rule, and (up to rounding) STV. We provide a self-standing proof in the Online Appendix.

[10] For precision, we further show that $n_R(y_R, \lambda_{\text{PAV}})$ is the unique integer such that

$$y_R \cdot (M+1) - 1 \leq n_R(y_R, \lambda_{\text{PAV}}) < y_R \cdot (M+1), \tag{2}$$

and use this value.

[11] Precinct-level results do not have a 1:1 map to census tracts. For each tract, the two-party vote share is generated as an average of its constituent precincts weighted by population and area overlap. About 80% of precincts are wholly contained in one census tract.

[12] Calculating accurate scores is a challenging task, but we note that the data set originates from a prominent company whose scores are used by many state and national parties, campaigns, and outside groups.

[13] These districts are drawn without partisan information and favor compact districts; however, the exact distribution is difficult to characterize given the nested hierarchical dependencies.

[14] This step for STV is enabled by Proposition 1. Because of the equivalence up to rounding, we use the same maps for PAV and STV, optimized for PAV.

[15] We note that variance of vote share (how much an area swings between parties across elections) is not the same as splitting ballots within an election, and so this sensitivity check is not exact. However, both variance across elections and splitting ballots within an election would cause vote leakage from a party.

[16] Selecting every other number starting at 10 and every third number starting at 20 is done for computational tractability.

[17] Formally, let $j = \lfloor \frac{N}{K} \rfloor$, and $L = N \bmod K$. Then, there are L districts of size $j+1$ and $K-L$ districts of size j. For example, when $N = 6$ and $K = 3$, $\{N_k\} = \{2,2,2\}$. When $N = 7$, $\{N_k\} = \{2,2,3\}$.

[18] Under our voting assumptions, for every PAV election with N_k seats, $\frac{1}{N_k+1}$ of the votes are wasted in total, and so the potential wasted votes for either party also goes to zero with district size.

[19] Whereas Duchin et al. (2019) find no SMD in Massachusetts with a majority of Republicans, we find it possible to draw such a district. The difference is data. They choose to include only presidential and senate elections, and their data is from a different time period; they find the effect in elections from 2000 to 2010. We also include statewide gubernatorial races in which Republicans have had more electoral success, and so our averaging concludes that Republicans compose 40% of the state. All redistricting approaches are sensitive to such data choices; results remain qualitatively the same though details for any given election may differ.

[20] In this section, we exclusively consider STV as it allows voters within a party to prioritize different candidates without risking their party overall representation by not approving a same-party candidate (as could happen with approval voting based methods such as Thiele rules). Studying approval methods requires analyzing primaries to select exactly N_k candidates from each party.

[21] See Section 205 here: https://beyer.house.gov/uploadedfiles/fair_representation_act_117th_final.pdf.

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